IOPscience iopscience.iop.org

Home Search Collections Journals About Contact us My IOPscience

Bose-Einstein condensation of an ideal Bose gas trapped in any dimension

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

1998 J. Phys. A: Math. Gen. 31 8289

(http://iopscience.iop.org/0305-4470/31/41/003)

View the table of contents for this issue, or go to the journal homepage for more

Download details:
IP Address: 171.66.16.104
The article was downloaded on 02/06/2010 at 07:17

Please note that terms and conditions apply.

J. Phys. A: Math. Gen. **31** (1998) 8289–8294. Printed in the UK PII: S0305-4470(98)93041-4

Bose–Einstein condensation of an ideal Bose gas trapped in any dimension

Lixuan Chen†‡, Zijun Yan†‡, Mingzhe Li‡ and Chuanhong Chen‡

† CCAST (World laboratory), PO Box 8730, Beijing, 100080, People's Republic of China
‡ Department of Physics, Xiamen University, Xiamen 361005, People's Republic of China§

Received 3 April 1998, in final form 14 July 1998

Abstract. The properties of a trapped ideal Bose gas in n-dimensional space are studied. General analytic expressions of the critical temperature T_c of Bose–Einstein condensation, the jump of heat capacity at T_c and the fraction of condensation at temperatures below T_c have been derived. How these physical quantities depend on the external potential, particle characteristics and space dimensionality are discussed. We find that when a proper external potential is taken, Bose–Einstein condensation may occur in any dimensional space.

1. Introduction

In an ideal Bose gas, the zero-momentum state can become macroscopically occupied, and the system then undergoes a phase transition—Bose–Einstein condensation (BEC) [1]. Owing to the development of techniques to trap and cool atoms, BEC was ultimately realized in 1995 [2–4]. These experimental achievements have stimulated great interest in the theoretical study of Bose gas.

Einstein's prediction is about ideal Bose gas. Although there are interactions between bosons and their effects on critical temperature T_c and fraction of condensation N_0/N are understood for the experimental situation, they seem to be a few per cent or less, when the density of the gas is low [5–7]. Moreover, due to the observation of Feshbach resonances [8], a tuning of the s-wave scattering length is possible and almost ideal Bose–Einstein condensates might be feasible. Thus it is well approximated that the Bose gas of low density is treated as an ideal gas.

That all observations of BEC have taken place in external potentials implies that an external potential has a significant effect on the performance of a Bose gas. In this paper, analytic expressions of some physical quantities about BEC are derived and used to discuss how BEC depends on the external potential, particle characteristics and space dimensionality. The results obtained here are quite general. Many main results, such as the results about free Bose gas, about BEC in one and two dimensions, about n-dimensional harmonic potential, and so on, in current literature and more new conclusions, such as the general criteria for the existance of BEC and for the behaviour (jump or not) of the specific heat, may be deduced from them.

§ Mailing address.

0305-4470/98/418289+06$19.50 © 1998 IOP Publishing Ltd

2. The total particle number and total energy of the system

We consider an ideal Bose gas in an external potential in n-dimensional space with a single-particle Hamiltonian

$$H = \varepsilon_0 \left(\frac{p}{p_0}\right)^s + U_0 \left(\frac{r}{r_0}\right)^t \tag{1}$$

where ε_0, U_0, s and t are all positive constants, p and r are, the momentum and coordinate respectively, of a particle; p_0 and r_0 are the characteristic momentum and coordinate, respectively. When the particle number of the system N is large enough and the energy-level spacing of the trapping potential is much smaller than $kT = \beta^{-1}$ (these two conditions are often satisfied, for example, in the experiment of the JILA group relative to BEC [2], $N \approx 2000$, when the frequency of the harmonic potential, $\omega = 2\pi \times 200/s$, is adopted, the critical temperature T_c is about 170 nK, one has $\hbar\omega/(kT) \approx 5.6 \times 10^{-3} \ll 1$), the Thomas–Fermi's semiclassical approximation is valid [9]. Thus, sums over quantum states may be replaced by integrals over phase space. The total number of particles N may then be expressed as

$$N = N_0 + N_e = N_0 + \sum \frac{g}{\mathrm{e}^{(H-\mu)/kT} - 1} = N_0 + \frac{g}{h^n} \int \frac{\mathrm{d}^n r \, \mathrm{d}^n p}{\mathrm{e}^{(H-\mu)/kT} - 1} \tag{2}$$

where N_0 and N_e are, respectively, the number of bosons in the ground state and in the excited states, μ is the chemical potential, k and h are, respectively, the Boltzmann and Plank constants, g is the spin degenerate factor. The volume of an n-dimensional sphere

$$V_n = C_n R^n = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2} + 1)} R^n \tag{3}$$

implies that

$$\mathrm{d}^n R = S_n(R) \, \mathrm{d}R = n C_n R^{n-1} \, \mathrm{d}R \tag{4}$$

where $S_n(R)$ is the surface of the n-dimensional sphere. By using equation (4) and the Bose integration

$$g_l(z) \equiv \frac{1}{\Gamma(l)} \int_0^\infty \frac{x^{l-1}}{z^{-1}\mathrm{e}^x - 1} \, \mathrm{d}x \tag{5}$$

equation (2) may be expressed as

$$N = N_0 + \frac{g C_n^2 \Gamma(\frac{n}{t} + 1) \Gamma(\frac{n}{s} + 1) (r_0 p_0)^n g_\lambda(z) (kT)^\lambda}{h^n U_0^{n/t} \varepsilon_0^{n/s}} \tag{6}$$

where

$$\lambda = \frac{n}{s} + \frac{n}{t} \tag{7}$$

$\Gamma(l) = \int_0^\infty y^{l-1} \mathrm{e}^{-y} \, \mathrm{d}y$ is the Gamma function, and $z = \exp(\mu/kT)$ is the fugacity. Along similar lines, the total energy E of the system can be written as

$$E = \int \frac{H \, \mathrm{d}^n r \, \mathrm{d}^n p}{\mathrm{e}^{(H-\mu)/kT} - 1} = \frac{g C_n^2 \, \lambda \Gamma(\frac{n}{t} + 1) \Gamma(\frac{n}{s} + 1) (r_0 p_0)^n g_{\lambda+1}(z) (kT)^{\lambda+1}}{h^n U_0^{n/t} \varepsilon_0^{n/s}}. \tag{8}$$

3. The critical temperature, fraction of condensation and jump of heat capacity

The chemical potential of a Bose gas cannot be positive and is monotonically increasing with temperature decreasing. When $T \to T_c$, there are $\mu \to 0$ and the particle number of the ground state is still macroscopically negligible, i.e. $N_0 = 0$. Then, we can obtain

$$kT_c = \left[\frac{Nh^n U_0^{n/t} \varepsilon_0^{n/s}}{gC_n^2 \Gamma(\frac{n}{t}+1)\Gamma(\frac{n}{s}+1)(r_0 p_0)^n \zeta(\lambda)} \right]^{1/\lambda} \tag{9}$$

from equation (6), where $\zeta(x) = g_x(1) = \sum_{J=1}^{\infty} \frac{1}{J^x}$ $(x \geqslant 1)$ is the Riemann zeta function. At a temperature T below T_c, from equations (6) and (9), we can obtain the fraction of condensation

$$\frac{N_0}{N} = 1 - \frac{N_e}{N} = 1 - \left(\frac{T}{T_c}\right)^{\lambda}. \tag{10}$$

From equation (8), the heat capacity with a given external potential at a temperature T above T_c is given by

$$C_{T>T_c} = \frac{\partial E_{T>T_c}}{\partial T} = Nk \left[\lambda(\lambda+1)\frac{g_{\lambda+1}(z)}{g_\lambda(z)} - \lambda^2 \frac{g_\lambda(z)}{g_{\lambda-1}(z)} \right] \tag{11}$$

where we have used $\frac{\partial g_{n+1}(z)}{\partial(\ln z)} = g_n(z)$ and $\frac{\partial N}{\partial T} = 0$. At a temperature T below T_c, there is $z = 1$ $(\mu = 0)$, so $g_{\lambda+1}(z) = \zeta(\lambda+1)$ and the total energy equation (8) reduces to

$$E_{T<T_c} = NkT\lambda \frac{\zeta(\lambda+1)}{\zeta(\lambda)} \left(\frac{T}{T_c}\right)^{\lambda}. \tag{12}$$

Consequently, at a temperature T below T_c, we have the heat capacity

$$C_{T<T_c} = Nk\lambda(\lambda+1)\frac{\zeta(\lambda+1)}{\zeta(\lambda)} \left(\frac{T}{T_c}\right)^{\lambda}. \tag{13}$$

Equations (11) and (13) give a jump of heat capacity at critical temperature

$$\Delta C_{T=T_c} = C_{T_c^-} - C_{T_c^+} = Nk\lambda^2 \frac{g_\lambda(1)}{g_{\lambda-1}(1)}. \tag{14}$$

4. Discussion

(1) Although a Bose gas only in an external potential has been studied, the above results can be used for a free Bose gas confined in a container. Because when $t \to \infty$, equation (1) describes a free Bose gas confined in a container with a radius r_0, that is, when $t \to \infty$, the potential $U \to \infty$ and $U \to 0$ in the regions of $r > r_0$ and $r < r_0$, respectively. Equations (9), (10) and (14) then become

$$T_c = \frac{\varepsilon_0}{k} \left[\frac{Nh^n}{gC_n^2 \Gamma(\frac{n}{s}+1)(r_0 p_0)^n \zeta(\frac{n}{s})} \right]^{s/n} \tag{15}$$

$$\frac{N_0}{N} = 1 - \left(\frac{T}{T_c}\right)^{n/s} \tag{16}$$

$$\Delta C_{T=T_c} = Nk \left(\frac{n}{s}\right)^2 \frac{\zeta(n/s)}{\zeta(n/s-1)}. \tag{17}$$

If we let $n = 3$, $g = 1$, $s = 2$ and $\varepsilon_0 = \frac{p_0^2}{2m}$ further, equations (15)–(17) may be reduced to

$$T_c = \frac{h^2}{2\pi mk}\left[\frac{N}{V\zeta\left(\frac{3}{2}\right)}\right]^{2/3} \tag{18}$$

$$\frac{N_0}{N} = 1 - \left(\frac{T}{T_c}\right)^{2/3} \tag{19}$$

$$\Delta C_{T=T_c} = 0. \tag{20}$$

Equations (18)–(20) describe a non-relativistic free ideal Bose gases in three-dimensional space, and coincide with the results in current textbooks of statistical mechanics [1], as they should.

(2) For the case of a non-relativistic spinless Bose gas trapped in a harmonic potential in three-dimensional space, $n = 3$, $g = 1$, $s = t = 2$ and $\varepsilon_0 = \frac{p_0^2}{2m}$, equations (9), (10) and (14) then give

$$T_c = \frac{h}{\pi k r_0}\left(\frac{U_0}{2m}\right)^{1/2}\left(\frac{N}{1.202}\right)^{1/3} \tag{21}$$

$$\frac{N_0}{N} = 1 - \left(\frac{T}{T_c}\right)^{3} \tag{22}$$

$$\frac{\Delta C_{T=T_c}}{Nk} = 6.58 \tag{23}$$

that have been obtained in [10]. It is worthwhile to point out that for an anisotropic harmonic osilator potential $U = \sum_{i=1}^{n}\frac{m}{2}\omega_i^2 r_i^2$, the results obtained here can also be used, because so long as we let $\omega_i r_i = \omega r_i^{'}$, the potential may be written in an isotropic form $U = \frac{m}{2}(\omega r^{'})^2$, where $(r^{'})^2 = \sum_{i=1}^{n}(r_i^{'})^2$.

Moreover, under a certain external potential, when n is chosen to be 2 or 1, the BEC in low-dimensional space, as discussed in [11], may be discussed with equations (9), (10) and (14). Furthermore, they may also be used to describe the case of fractional dimensionality, and agree with [12].

(3) The results obtained in this paper are not only the general forms of some important conclusions obtained in current literature as mentioned above, but also may be used to derive some novel and general conclusions about BEC. For example, the general criterion for BEC occurence

$$\lambda = \frac{n}{s} + \frac{n}{t} > 1 \tag{24}$$

can be obtained from equation (9). That is, only when equation (24) is satisfied, may BEC take place. Equation (24) mirrors that the criterion relates not only to the dimensionality of space and characteristics of particles, but also to the shape (not the strength) of the external potential.

Considering non-relativistic Bose gas (i.e. $s = 2$), it is well understood that there exists no BEC for a free system (i.e. $t \to \infty$) if the space dimensionality $n \leqslant 2$. In contrast, when the system is trapped in a harmonic potential (i.e. $t = 2$), it may undergo BEC if $n = 2$, however, there still exists no BEC if $n = 1$ [11]. But if another shape of external potential, e.g. $t = \frac{1}{2}$, is adopted, the non-relativistic Bose gas may undergo BEC even in one-dimensional space. From equation (24), we can draw a conclusion that BEC may take place in any dimensional space if a proper external potential is introduced.

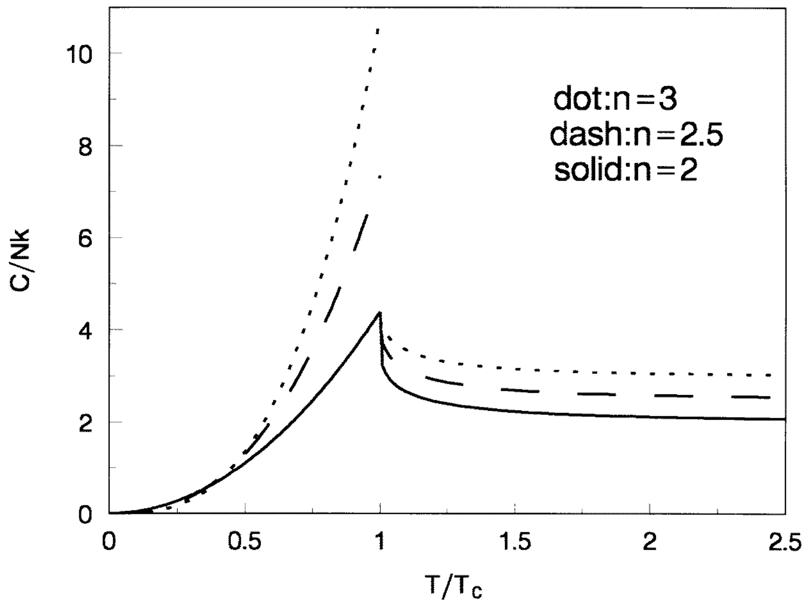

Figure 1. The heat capacity versus temperature for non-relativistic ideal Bose gas trapped in a harmonic potential when the space dimensionality $n = 3, 2.5$, and 2.

For an ultrarelativistic Bose gas (i.e. $s = 1$), there is only one case that exists in which there is no BEC, i.e. when the system is free and in one-dimensional space. Any other case with external potential (i.e. $t \not\to \infty$) or $n > 1$ may undergo BEC. It is different from [13] which deals only with a free system. Of course, when the inclusion of antiparticles in relativistic theories is considered, the conclusion should be reconsidered.

(4) Equation (14) implies a criterion on the continuity of heat capacity at the critical temperature. If

$$\lambda = \frac{n}{s} + \frac{n}{t} > 2 \tag{25}$$

there is a jump of heat capacity at critical temperature, otherwise, there exists no jump if $1 < \lambda \leqslant 2$ or even no BEC if $\lambda \leqslant 1$. Therefore, for a non-relativistic ideal Bose gas trapped in a harmonic potential in three-dimensional space, there is a jump of heat capacity at critical temperature given by equation (14). For a non-relativistic free ideal Bose gas in three-dimensional space, $s = 2$, $n = 3$ and $t \to \infty$, the heat capacity is continuous at critical temperature because equation (24) is satisfied while equation (25) is not.

Figure 1 shows the heat capacity versus temperature for a non-relativistic ideal Bose gas trapped in a harmonic potential when the space dimensionality is, respectively, 3, 2.5 and 2.

(5) Equation (9) mirrors the dependence of critical temperature on the total particle number, shape of external potential, characteristics of particles and dimensionality of space. For the case of a spinless non-relativistic ideal Bose gas trapped in a harmonic potential [14], $U_0 = \frac{m\omega^2}{2} r_0^2$, $\varepsilon_0 = \frac{p_0^s}{2m}$, and equation (9) then reduces to

$$T_c = \frac{\hbar\omega}{k} \left[\frac{N}{\zeta(n)} \right]^{1/n}. \tag{26}$$

Equation (26) expresses the dependence of critical temperature on the space dimensionality.

After a simple calculation, we have values of $\frac{kT_c}{\hbar\omega}$ at $0.94N^{\frac{1}{3}}$, $0.89N^{\frac{2}{5}}$ and $0.78N^{\frac{1}{2}}$ when the space dimensionality is, respectively, 3, 2.5 and 2.

Equation (9) can also be reduced to discuss the dependence of critical temperature on the external potential for an ideal Bose gas in a certain dimensional space. The possibility of controlling the critical temperature through setting up some parameters in equation (9), especially the shape of external potential, to facilitate the realization of BEC is exciting.

References

[1] Huang K 1987 *Statistical Mechanics* 2nd edn (New York: Wiley)
[2] Anderson M H, Ensher J R, Matthews M R, Wieman C E and Cornell E A 1995 *Science* **269** 198
[3] Bradley C C, Sackett C A, Tollett J J and Hulet R G 1995 *Phys. Rev. Lett.* **75** 1687
[4] Davis K B, Mewes M-O, Andrew M R, van Druten N J, Durfee D S, Kurn D M and Ketterle W 1995 *Phys. Rev. Lett.* **75** 3969
[5] Zheng J, Yan Z, Lin Z and Chen L 1997 *Chin. J. Comput. Phys.* **14** 690
[6] Giorgini S, Pitaevskii L P and Stringari S 1997 *J. Low Temp. Phys.* **109** 309
[7] Ensher J R, Jin D S, Matthews M R, Wieman C E and Cornell 1996 *Phys. Rev. Lett.* **77** 4984
[8] Inouye S, Andrews M R, Stenger J, Miesner H J, Stamper-Kurn D M and Ketterle W 1998 *Nature* **392** 151
[9] Chou T T, Yang C N and Yu L H 1997 *Phys. Rev.* A **53** 4257
[10] Bagnato V, Pritchard D E and Kleppner D 1987 *Phys. Rev.* A **35** 4554
[11] Bagnato V and Kleppner D 1991 *Phys. Rev.* A **44** 7439
[12] Kim Sang-Hoon 1997 *Int. J. Bifurcation Chaos Appl. Sci. Eng.* **7** 1053
[13] Beckmann R, Karsch F and Miller D E 1979 *Phys. Rev. Lett.* **43** 1277
[14] Kirston K and Toms J 1998 *Phys. Lett.* **243A** 137